J. Differential Equations 246 (2009) 1500–1522

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Nonlinear BVPS with functional parameters

Aleksandra Orpel

Faculty of Mathematics and Computer Science, University of Lodz, Banacha 22, 90-238 Lodz, Poland

A R T I C L E   I N F O

Article history:
Received 29 May 2007
Available online 6 December 2008

MSC:
primary 35J65, 34B15
secondary 49J40, 49J60

Keywords:
Nonlinear elliptic problems
Positive solutions
Radial solution
Duality method
Variational principle

A B S T R A C T

We study the existence, nonexistence and properties of solutions for a certain class of second-order ODEs and their dependence on functional parameters, also in the case when nonlinearities are, in some sense, singular. This approach is based on variational methods and cover both sublinear and superlinear cases. We develop a duality theory and variational principles for this problem. As a consequence of the duality theory we give a numerical version of the variational principle which enables approximation of the solution for our problem. We apply these results to obtain the existence of bounded, radial and positive classical solutions for the BVP of elliptic type. Observe that our method allows us to investigate a certain class of elliptic systems in both bounded annular domain and exterior domain.

1. Introduction

Many problems modeled by systems of elliptic PDEs arise in various areas of applied mathematics, in biological, chemical or physical phenomenas, for example in thermal ignition of gases, in theory of nonlinear diffusion generated by nonlinear sources (see e.g. [6–8,12] and references therein). Recently the research on positive (nonnegative) solutions of the systems of nonlinear equations containing Laplacian or perturbed p-Laplace operator has been very active and enjoying of increasing interest (see e.g. [1,2,11,13,17–19] and references therein).

We also want to join in the discussion concerning systems of BVPs of elliptic type, especially we are interested in the dependence of solutions on functional parameters. We are going to work in two areas. Our first aim is to investigate the behavior of solutions of the problems including perturbed p-Laplace operator provided that parameters are convergent in some functional space. It appears that quite weak and local only assumptions made on nonlinearities are still sufficient to prove that solutions depend continuously (in some sense) on parameters. As a consequence of this result we obtain

E-mail address: orpela@math.uni.lodz.pl.

doi:10.1016/j.jde.2008.10.030

the continuous dependence of solutions on boundary conditions. The other aspect of this work is associated with the dependence of number of solutions on parameters. It is worth noting that we discuss the case when the nonlinearities are not necessarily linear with respect to the parameter as in many papers (see e.g. [5,14]), where the generalized eigenvalue problem is discussed. We want to present the methods which allow us to study the systems of elliptic equations in both annular domain (as in (1) given below) and exterior domain (as in (2) given below). Moreover due to the fact that we do not need any information concerning behavior of nonlinearity at infinity, we cover both sub- and superlinear problems. To these effects we base ourselves on dual variational approach enriched by some elements coming from topological methods. This topological origin of some of our ideas is described in Section 2 devoted to the existence results.

Let us recall some results concerning similar problems. In the case when the domain Ω is a ball in R^n and the nonlinearities are positive, the existence of positive radial solutions and their properties were studied in [1] by the method of topological degree. In paper [2] the local and global behavior of solutions in unbounded domain Ω for the elliptic systems is discussed provided that nonlinearities satisfy some conditions of polynomial type. Semilinear elliptic systems with singularities are investigated in [11], where the authors formulate necessary conditions for the existence of bounded positive solutions in the case when the principal part is the Laplace operator. A.E. Khalil, M. Ouanan and A. Touzani (in [13]) used variational methods to prove the existence and regularities of solutions for elliptic systems with nonlinearities satisfying some growth conditions of exponential type. Joao Marcos de Ó, S. Lorca, J. Sanchez and P. Ubilla apply topological methods like fixed point theorem due to Krasnoselskii, fixed point index theory and upper–lower solutions methods to investigate elliptic systems. They discuss the existence, nonexistence and multiplicity of solutions with radial symmetry for elliptic systems in annular domains or exterior domains similar to the following

$$\begin{gathered}
-\Delta u = f_1\big(\|y\|, u, v\big) \quad \text{for } r < \|y\| < R \text{ with } y \in R^n, \\
-\Delta v = f_2\big(\|y\|, u, v\big) \quad \text{for } r < \|y\| < R \text{ with } y \in R^n, \\
u = v = 0 \quad \text{for } \|y\| = r \text{ with } y \in R^n, \\
u = a \quad \text{and} \quad v = b \quad \text{for } \|y\| = R \text{ with } y \in R^n
\end{gathered} \tag{1}$$

and/or

$$\begin{gathered}
-\Delta u = f_1\big(\|y\|, u, v\big) \quad \text{for } 1 < \|y\| \text{ with } y \in R^n, \\
-\Delta v = f_2\big(\|y\|, u, v\big) \quad \text{for } 1 < \|y\| \text{ with } y \in R^n, \\
u = a \quad \text{and} \quad v = b \quad \text{for } \|y\| = 1 \text{ with } y \in R^n, \\
\lim_{\|y\|\to\infty} u(y) = 0 \quad \text{and} \quad \lim_{\|y\|\to\infty} v(y) = 0,
\end{gathered} \tag{2}$$

where a, b are given nonnegative real numbers, $\|y\| = \sqrt{\sum_{i=1}^k y_i^2}$ (see e.g. [17–20] and references therein).

It is well-known fact that the investigation of existence of radial solutions for (1) or (2) leads, via suitable substitution, to the systems of Dirichlet problems governed by ordinary equations of the form

$$\begin{cases} -x_i''(t) = g_i\big(t, \mathbf{x}(t), a, b\big) \quad \text{on } (0,1), \\ x_i(0) = x_i(1) = 0, \end{cases} \quad \text{for all } i \in \{1,2\}, \tag{3}$$

where nonlinearities may even be, in some sense, singular (see (2)).

Problems similar to (2) (or (3)) has been also widely discussed by many authors for single equation (see e.g. [3,4,9,10,14,15,24] and references therein). The main tools used by the authors are based on the fixed point theorems in cones due to Krasnoselskii (in [23,26]) and perturbation method together with some fixed point theorems which follow from Leray–Schauder degree theory [25]. The authors required, among others, that the nonlinearity $f:(1,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous and satisfies some extra conditions concerning its behavior at zero and/or at infinity. For example, in [26] one assumed, among others, that there exists a set B of positive measure such that

$$\lim_{u\to\infty}\frac{f(s,u)}{u}=+\infty \quad \text{uniformly w.r.t. } s\in B \tag{B2}$$

and there exists a function $p\in C((1,\infty))$, $\int_1^\infty s(1-s^{2-n})p(s)\,ds<+\infty$, such that

$$\lim_{u\to 0^+}\frac{f(s,u)}{p(s)u}=0 \quad \text{uniformly w.r.t. } s\in(1,+\infty). \tag{B3}$$

It is worth noting that we study system (5) without conditions similar to (B2) or (B3) often met also in other papers.

The first object of this paper is to consider situation when the differential operator is defined by one-dimensional perturbed q_i*-Laplacian* for each $i\in\{1,\dots,k\}$. Since we will base ourselves on variational methods we discuss the case when nonlinearities are the partial derivatives of a certain real-valued function, namely when there exists $\widetilde{G}:(0,1)\times R^k\times R^s\to R$, such that $\widetilde{G}'_i=\frac{\partial\widetilde{G}}{\partial x_i}=g_i$, for all $i\in\{1,\dots,k\}$. Thus, our aim is to answer the question when the following systems of Dirichlet problems

$$\begin{cases}-\big(l_i(t)\big|x_i'(t)\big|^{q_i-2}x_i'(t)\big)'=\widetilde{G}'_i\big(t,\mathbf{x}(t),\mathbf{z}(\mathbf{t})\big) & \text{a.e. in } (0,1),\\ x_i(0)=x_i(1)=0, \end{cases} \quad \text{for all } i\in\{1,\dots,k\}, \tag{4}$$

where $k\in\mathbf{N}:=\{1,2,\dots\}$, $\mathbf{q}:=(q_1,\dots,q_k)$, $q_i\in\mathbf{N}\setminus\{1\}$ and are even, possesses a solution $\mathbf{x}=(x_1,\dots,x_k)$ lying in a certain k-dimensional interval of the product of nonnegative axes, for fixed parameter $\mathbf{z}$. Precisely, by a solution of (4) we understand a function $\mathbf{x}\in C([0,1],R^k)\cap C^1((0,1),R^k)$ such that $[l_i|x_i'|^{q_i-2}x_i']_{i=1}^k\in W^{1,\mathbf{q}'}((0,1),R^k):=W^{1,q_1'}((0,1),R)\times\cdots\times W^{1,q_k'}((0,1),R)$, where $q_i':=q_i/(1-q_i)$ for $i\in\{1,\dots,k\}$, and $\mathbf{x}$ satisfies (4). We show that if $\widetilde{G}'_i\in C((0,1)\times\mathbf{J}\times R^s)$ and $q_i=2$, for all $i\in\{1,\dots,k\}$, $\mathbf{x}$ is also a classical solution of (4), namely $\mathbf{x}\in C([0,1],R^k)\cap C^2((0,1),R^k)$.

Our research covers the case when the nonlinearity depends on functional parameter $\mathbf{z}:(0,1)\to R^s$, $s\in\mathbf{N}$, $\mathbf{z}\in\mathbf{Z}$, where $\mathbf{Z}\subset L^p(0,1)$. We also characterize parameters for which there are no positive solutions of (4).

As we mentioned above another purpose of our work is to study the continuous (in some sense) dependence of solutions of (4) on functional parameters under only local conditions made on the nonlinearities. Precisely, we show that the sequence of positive solutions (up to a subsequence) tends uniformly in $[0,1]$ provided that the sequence of parameters is convergent a.e. in $(0,1)$. Applying this result we formulate the conclusions concerning continuous dependence on boundary conditions.

2. The existence results for ODEs

In this section we shall take up the following system of ODEs

$$\begin{cases}-\big(l_i(t)\big|x_i'(t)\big|^{q_i-2}x_i'(t)\big)'=G'_i\big(t,\mathbf{x}(t)\big) & \text{a.e. in } (0,1),\\ x_i(0)=x_i(1)=0, \end{cases} \quad \text{for all } i\in\{1,\dots,k\}. \tag{5}$$

It is due to the fact that we start with the studying the existence results for positive solutions for (4) with $\mathbf{z}$ fixed. Thus, for the simplicity of notations, we consider (5) instead of (4). We propose an approach based on the following assumptions

(G1) there exist k-dimensional interval $\mathbf{I} := [0, \tilde{d}_1] \times \cdots \times [0, \tilde{d}_k]$, $\tilde{d}_1, \ldots, \tilde{d}_k \in R_+ := (0, +\infty)$, and a neighborhood $\mathbf{J}$ of $\mathbf{I}$ such that $G : (0,1) \times \mathbf{J} \to R$, $G \in C((0,1) \times \mathbf{J})$, is convex with respect to the second variable in $\mathbf{J}$, and $|\int_0^1 G(l, \mathbf{0})\, dl| < +\infty$;

(G2) for each $i \in \{1, \ldots, k\}$, partial derivative $G_i' = \frac{\partial G}{\partial x_i}$ exists and is nonnegative in $(0,1) \times \mathbf{J}$, $(0,1) \ni t \mapsto \sup_{\tilde{\mathbf{x}} \in \mathbf{I}} G_i'(t, \tilde{\mathbf{x}}) \in L^{q_i'}((0,1), R)$, *and* $\int_0^1 G_i'(l, \mathbf{0})\, dl \neq 0$;

(G3) there exist $l_{\min}, l_{\max} > 0$ such that for each $i \in \{1, \ldots, k\}$, $l_i \in C([0,1], R) \cap C^1((0,1), R)$ and $l_{\min} \leqslant l_i(t) \leqslant l_{\max}$ on $[0,1]$.

We plan to treat (5) with variational methods, so we start with definition of the action functional associated with our system. Let $J : A_0^k \to R$ be given as

$$J(\mathbf{x}) = \sum_{i=1}^{k} \int_0^1 \frac{l_i(t)}{q_i} \left|x_i'(t)\right|^{q_i} dt - \int_0^1 G\big(t, \mathbf{x}(t)\big)\, dt, \tag{6}$$

where A_0^k is the space of continuous functions $\mathbf{x} \in C^1((0,1), R^k) \cap C([0,1], R^k)$ with $\mathbf{x}' \in \mathbf{L} := L^{q_1}((0,1), R) \times \cdots \times L^{q_k}((0,1), R)$ and $\mathbf{x}(0) = \mathbf{x}(1) = \mathbf{0}$, $\mathbf{0} := (0, \ldots, 0) \in R^k$, with the norm $\|\mathbf{x}\|_{A_0^k} = (\sum_{i=1}^k \int_0^1 |x_i'(t)|^{q_i}\, dt)^{1/q_i}$.

Conditions (G1)–(G3) describe the case when the classical methods of calculus of variations cannot be applied in the standard way. First of all J is not necessarily bounded below (above) in its natural domain A_0^k, so that we must look for critical points of (6) of "minmax" type or find subsets W and W^d, on which the action functional J or the dual one – J_D – is bounded. Of course, we have the Morse theory and its generalization, the saddle points theorems, the mountain pass theorems, but all these methods do not exhaust all critical points of J. We shall apply the approach based on the special definitions of sets W and W^d over which we will calculate minimum of J and J_D. In this case, (5) cannot be deduced directly from the existence of a minimum of J in W, since W is not the whole space and then the principle of minimal action does not work. Thus we have to find links between arguments of minimum of J in W and the solutions of (5). To this effect we shall develop duality and just because of this theory we are able to omit in our proof of the existence of critical points the deformation lemmas, the Ekeland variational principle or PS type conditions. Since W and W^d are not linear subspaces, the question is how to calculate the conjugate of functionals over nonlinear spaces. To overcome these difficulties we must characterize W (as in [16,21,22]) by a certain property which plays the crucial role in a duality theory developed here. Due to the duality we prove the relations between the infimum of the energy functional on W associated with the problem and the infimum of the dual functional on a corresponding set W^d. Finally, we show that the relations between minimizers of both functionals imply a variational principle and, in consequence, their links to the solutions of our system.

To construct the set W of arguments of J we need the following set

$$\begin{aligned}\overline{W} = \big\{&\mathbf{x} = (x_1, \ldots, x_k) \in A_0^k,\ x_i(t) \geqslant 0 \text{ for all } t \in [0,1] \text{ and } i \in \{1, \ldots, k\} \\ &\text{and } \big[l_i \big|x_i'\big|^{q_i - 2} x_i'\big]_{i=1}^k \in W^{1,\mathbf{q}'}\big((0,1), R^k\big)\big\}.\end{aligned}$$

Definition 1. We say that nonempty set $W \subset \overline{W}$ has property (D) if for each $\mathbf{x} = (x_1, \ldots, x_k) \in W$ there exists $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_k) \in W$ such that for all $i \in \{1, \ldots, k\}$

$$\begin{gathered}-\big(l_i(t)\big|\tilde{x}_i'(t)\big|^{q_i - 2} \tilde{x}_i'(t)\big)' = G_i'\big(t, \mathbf{x}(t)\big) \quad \text{a.e. in } (0,1), \\ \tilde{x}_i(0) = \tilde{x}_i(1) = 0.\end{gathered} \tag{7}$$

Remark 1. Let us note that for all $\mathbf{x} \in W$ such that $\mathbf{x}(t) \in I$ in $(0,1)$, (7) is equivalent to

$$-\left(\left(l_1(t)\left|\tilde{x}_1'(t)\right|^{q_1-2}\tilde{x}_1'(t)\right)', \ldots, \left(l_k(t)\left|\tilde{x}_k'(t)\right|^{q_k-2}\tilde{x}_k'(t)\right)'\right) \in \partial G\left(t, \mathbf{x}(t)\right) \quad \text{a.e. in } (0,1),$$

where

$$\partial G(t,\mathbf{z}) := \left\{\mathbf{u}^* \in R^k,\ G(t,\mathbf{w}) \geqslant G(t,\mathbf{z}) + \left\langle \mathbf{u}^* | (\mathbf{w}-\mathbf{z})\right\rangle \text{ for all } \mathbf{w} \in R^k\right\} \tag{8}$$

for all $\mathbf{z} = (z_1, \ldots, z_k) \in R^k$ and $t \in (0,1)$, with $\langle \mathbf{a}|\mathbf{b}\rangle := \sum_{i=1}^k a_i b_i$ for all $\mathbf{a} = (a_1, \ldots, a_k)$, $\mathbf{b} = (b_1, \ldots, b_k) \in R^k$.

It is also worth noting that property (D) is equivalent to the fact that W is an invariant set for a certain integral operator that will be discussed later.

Now we assume additionally that

(S) there exists a nonempty set $W \subset \overline{W}$ satisfying the following conditions:
- (S1) W has property (D);
- (S2) for each $\mathbf{x} \in W$, $\mathbf{x}(t) \in \mathbf{I}$ for all $t \in [0,1]$.

Our task is now to show that problem (5) possesses at least one solution in each W satisfying (S1)–(S2).

2.1. Examples of sets with property (D)

In this section we formulate conditions which allow us to give an explicit definition of $W \subset \overline{W}$ satisfying (S1)–(S2). To this effect let us consider the situation when (G1)–(G3) are valid. Now we will show topological roots of the property (D), which is crucial in our investigation. It is well-known fact that the first step of methods based on the fixed point theorems is to find an invariant set for a certain operator. For the simplicity of calculation we assume $q_i = 2$ for all $i \in \{1, \ldots, k\}$. Let us define the operator $\mathbf{A}$ as follows: $\mathbf{Ax} := (A_1\mathbf{x}, \ldots, A_k\mathbf{x})$, where

$$A_i\mathbf{x}(t) = \int_0^1 \mathbf{G}_i(s,t) G_i'\left(s, \mathbf{x}(s)\right) ds, \tag{9}$$

with the Green's function $\mathbf{G}_i$ associated with a linear homogeneous problem corresponding to ith equation of (5) is given by

$$\mathbf{G}_i(s,t) := \frac{1}{c_i}\begin{cases} \int_0^s h_i(r)\,dr \int_t^1 h_i(r)\,dr & \text{for } 0 \leqslant s \leqslant t, \\ \int_0^t h_i(r)\,dr \int_s^1 h_i(r)\,dr & \text{for } t < s \leqslant 1, \end{cases}$$

where $c_i := \int_0^1 h_i(r)\,dr$ and $h_i(s) = (\frac{1}{l_i(s)})$ in $(0,1)$. If we assume additionally that $l_i \equiv 1$ for all $i \in \{1, \ldots, k\}$, $\mathbf{G}_i$ can be rewritten as

$$\mathbf{G}_i(s,t) := \begin{cases} s(1-t) & \text{for } 0 \leqslant s \leqslant t, \\ t(1-s) & \text{for } t < s \leqslant 1. \end{cases}$$

It is clear that in order to find a solution of (5) it suffices to find a fixed point for $\mathbf{A}$ in some set. So if we investigate the case when $G_i' \in C((0,1) \times \mathbf{J})$ and want to apply e.g. Schauder's fixed point theorem, the fixed point theorem on cones due to Krasnoselskii or any other fixed point theorem, we have to find an invariant set $\mathbf{X}$ for operator $\mathbf{A}$, namely $\mathbf{X}$ such that $\mathbf{AX} \subset \mathbf{X}$. This inclusion means that for each $\mathbf{x} = (x_1, \dots, x_k) \in \mathbf{X}$ there exists $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_k) \in \mathbf{X}$ such that $\mathbf{Ax} = \tilde{\mathbf{x}}$, that is

$$\tilde{x}_i(t) = \int_0^1 \mathbf{G}_i(s,t) G_i'\big(s, \mathbf{x}(s)\big)\, ds \quad \text{on } (0,1), \tag{10}$$

for each $i \in \{1, \dots, k\}$. (10) can be rewritten as (7) and leads to the conclusion that $\mathbf{X}$ has property (D).

Lemma 1. *Suppose that conditions* (G1)–(G3) *hold and assume additionally that*

(G4) *there exists* $0 < d \leqslant \min_{i \in \{1,\dots,k\}} \tilde{d}_i$ *such that for all* $\mathbf{u} \in \{\mathbf{u} \in R^k;\ \|\mathbf{u}\| \leqslant d\}$ *and all* $i \in \{1, \dots, k\}$

$$\int_0^1 G_i'(s, \mathbf{u})\, ds \leqslant 4dc_i l_{\min}^2.$$

Then for the set W_1 *defined as follows*

$$W_1 = \big\{\mathbf{x} \in \overline{W},\ \|\mathbf{x}\|_\infty \leqslant d\big\},$$

with $\|\mathbf{x}\|_\infty = \sup_{t \in [0,1]} \|\mathbf{x}(t)\|$, *conditions* (S1) *and* (S2) *hold.*

Proof. First we show that W_1 has property (D). Fix $\mathbf{x} = (x_1, \dots, x_k) \in W$. It is clear that $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_k)$ with $\tilde{x}_i = A_i \mathbf{x}$ on $(0,1)$ and A_i given by (9), belongs to $C([0,1], R^k) \cap C^1((0,1), R^k)$, and $\tilde{x}_i \geqslant 0$ on $[0,1]$. Moreover, (G4) guarantees the following estimate for all $t \in [0,1]$

$$\begin{aligned}
\tilde{x}_i(t) &= A_i \mathbf{x}(t) \\
&= \frac{1}{c_i} \int_t^1 h_i(r)\, dr \int_0^t \int_0^s h_i(r)\, dr\, G_i'\big(s, \mathbf{x}(s)\big)\, ds + \frac{1}{c_i} \int_0^t h_i(r)\, dr \int_t^1 \int_s^1 h_i(r)\, dr\, G_i'\big(s, \mathbf{x}(s)\big)\, ds \\
&\leqslant \frac{1}{c_i l_{\min}^2} \left((1-t) \int_0^t s G_i'\big(s, \mathbf{x}(s)\big)\, ds + t \int_t^1 (1-s) G_i'\big(s, \mathbf{x}(s)\big)\, ds \right) \\
&\leqslant \frac{1}{c_i l_{\min}^2} (1-t)t \left(\int_0^1 G_i'\big(s, \mathbf{x}(s)\big)\, ds \right) \leqslant \frac{1}{4c_i l_{\min}^2} \int_0^1 G_i'\big(s, \mathbf{x}(s)\big)\, ds \leqslant d.
\end{aligned}$$

Moreover, by the properties of G and the fact that $\mathbf{x} \in W$, we state, by the above definition of $\tilde{x}_i$, that $l_i \tilde{x}_i' \in W^{1,2}((0,1), R)$. Thus $\tilde{\mathbf{x}} \in W_1$, what we have claimed. As a consequence of the above estimate we get condition (S2). □

In the last section we will give explicit examples of elliptic system similar to (2) which leads to system of ODEs with nonsmooth (or not defined in product of whole positive half-lines) nonlinearities satisfying (G1)–(G3) and (G4). As we shall show, our results can be applied for both sublinear and superlinear cases.

We have to mention that in the case described by assumptions (G1)–(G4) we can apply Schauder's fixed point theorem to obtain the existence of solution $\bar{\mathbf{x}}$ of (5) from W_1 as a fixed point of operator $\mathbf{A}$ if we assume additionally that $G_i' \in C((0,1) \times \mathbf{J})$. It follows from the convexity of W_1, inclusion $\mathbf{A}W_1 \subset$

W_1 and the fact that $\mathbf{A}$ is the completely continuous operator mapping $B := \{C([0,1], R^k);\ \|\mathbf{x}\|_\infty \leqslant d\}$ into B.

Let us note that in our approach the convexity of W is not necessary. Now we want to obtain more information concerning solutions of our problem. We look for positive lower estimate of the solution. To this end we consider a subset W_2 of $\overline{W}$ whose elements are bounded also below by common nonzero function.

Lemma 2. *Let us assume that* (G1)–(G4) *hold and*

(G5) *there exist a set $C \subset (0,1)$ of positive measure and $\mathbf{u} = (u_1, \dots, u_k) \in \overline{W}$, $\mathbf{u}(\mathbf{t}) \in \mathbf{J} \cap R^k_+$ in $(0,1)$ such that for all $i \in \{1, \dots, k\}$, $A_i\mathbf{u}(t) \geqslant u_i(t)$ in C and*

$$\inf_{\mathbf{x} \in \mathbf{I}_i(s)} G'_i(s, \mathbf{x}) \geqslant G'_i\bigl(s, \mathbf{u}(s)\bigr) \quad \text{a.e. in } (0,1),$$

where $\mathbf{I}_i(t) := \{\mathbf{z} = (z_1, \dots, z_k) \in R^k, z_i > u_i(t)\}$ for all $t \in (0,1)$.

Then the set

$$W_2 := W_1 \cap \bigl\{\mathbf{x} \in \overline{W},\ \text{there exists } i \in \{1, \dots, k\} \text{ such that } u_i(t) \leqslant x_i(t) \text{ in } C\bigr\},$$

where W_1 is given in Lemma 1, *satisfies conditions* (S1)–(S2).

Proof. Fix $\mathbf{x} \in W_2$. By Lemma 1, $\tilde{\mathbf{x}} := \mathbf{A}\mathbf{x}$ belongs to W_1. We denote by i_0 an element of $\{1, \dots, k\}$ such that $u_{i_0}(t) \leqslant x_{i_0}(t)$ *in* C. So now it suffices to show that $u_{i_0}(t) \leqslant \tilde{x}_{i_0}(t)$ in C. To this end we calculate for each $t \in C$

$$\begin{aligned}
\tilde{x}_{i_0}(t) &= A_{i_0}\mathbf{x}(t) \\
&= \frac{1}{c_{i_0}} \int_t^1 h_{i_0}(r)\,dr \int_0^t \int_0^s h_{i_0}(s)\,dr\, G'_{i_0}\bigl(s, \mathbf{x}(s)\bigr)\,ds + \frac{1}{c_{i_0}} \int_0^t h_{i_0}(r)\,dr \int_t^1 \int_s^1 h_{i_0}(r)\,dr\, G'_{i_0}\bigl(s, \mathbf{x}(s)\bigr)\,ds \\
&\geqslant \frac{1}{c_{i_0}} \int_t^1 h_{i_0}(r)\,dr \int_0^t \int_0^s h_{i_0}(s)\,dr\, G'_{i_0}\bigl(s, \mathbf{u}(s)\bigr)\,ds + \frac{1}{c_{i_0}} \int_0^t h_{i_0}(r)\,dr \int_t^1 \int_s^1 h_{i_0}(r)\,dr\, G'_{i_0}\bigl(s, \mathbf{u}(s)\bigr)\,ds \\
&= A_{i_0}\mathbf{u}(t) \geqslant u_{i_0}(t).
\end{aligned}$$

Summarizing $\tilde{\mathbf{x}} \in W_2$. Condition (S2) is a simply consequence of the fact that $\tilde{\mathbf{x}} \in W_1$. $\square$

In this case we cannot apply Schauder's fixed point theorem to obtain the existence of solutions in W_2, since W_2 is not convex.

2.2. Basic properties of solutions

In this subsection we assume that (G1)–(G3) hold.

Proposition 3. *If $\mathbf{x} = (x_1, \dots, x_k)$ is a solution of* (5) *such that $\mathbf{x}(t) \in \mathbf{I}$ for all $t \in [0,1]$, then for all $i \in \{1, \dots, k\}$*

(i) *$B_i := \{t \in (0,1), x'_i(t) = 0\}$ is a nonempty interval*;
(ii) *$x_i(t) > 0$ for all $t \in (0,1)$*;
(iii) *$0 < \inf B_i \leqslant \sup B_i < 1$*;
(iv) *x_i is strictly increasing in $[0, I_i]$ and x_i is strictly decreasing in $[S_i, 1]$ consequently, by* (i), *for all $t \in B_i$, $x_i(t) = \max_{l \in [0,1]} x_i(l)$.*

Proof. Fix $i \in \{1, \dots, k\}$. Taking into account assumption (G2), x_i is not identically equal to zero in $[0,1]$.

(i) Let us define $k_i : (0,1) \to R$ as follows $k_i(t) = l(t)|x_i'(t)|^{q-2}x_i'(t)$ for $t \in (0,1)$. Since x_i is a solution of (5), we get $k_i \in C((0,1), R)$ and $k_i'(t) = -G_i'(t, \mathbf{x}(t)) \leqslant 0$ a.e. in $(0,1)$. Therefore k_i is nonincreasing in $(0,1)$. Taking into account the fact that $x_i \in C^1((0,1), R) \cap C([0,1], R)$ and $x_i(0) = x_i(1) = 0$, the Rolle's theorem leads to the existence of $t_0 \in (0,1)$ such that $x_i'(t_0) = 0$, namely $t_0 \in B_i$. Let us take t_1, $t_2 \in B_i$, without lost of generality, we may assume that $t_1 < t_2$. The definition of k_i implies equalities $k_i(t_1) = k_i(t_2) = 0$, and consequently, $k_i(t) = 0$ for all $t \in [t_1, t_2]$, where the last assertion is due to the monotonicity of k_i. Thus, $x_i'(t) = 0$ for all $t \in [t_1, t_2]$, so $[t_1, t_2] \subset B_i$. Finally, we state that B_i is an interval.

(ii) Coming to the second part of the proposition we suppose otherwise and assume the existence of $t_0 \in (0,1)$ such that $x_i(t_0) = 0$. Then, by the Rolle's theorem, there exist $a_1 \in (0, t_0)$ and $b_1 \in (t_0, 1)$ such that $x_i'(a_1) = x_i'(b_1) = 0$. Thus $a_1, b_1 \in B_i$ and, by (i), for all $t \in [a_1, b_1]$, $x_i'(t) = 0$, consequently $x_i \equiv const$ in $[a_1, b_1]$. Since $t_0 \in [a_1, b_1]$, $x_i \equiv 0$ in $[a_1, b_1]$. Now we proceed with this process. So, when we have $a_n \in (0, a_{n-1})$ and $b_n \in (b_{n-1}, 1)$ such that $x_i \equiv 0$ in $[a_n, b_n]$, we infer the existence of $a_{n+1} \in (0, a_n)$ and $b_{n+1} \in (b_n, 1)$ such that $x_i \equiv 0$ in $[a_{n+1}, b_{n+1}]$. Let us note that $\{a_n\}_{n=1}^{\infty}$ is decreasing and bounded below by 0 and $\{b_n\}_{n=1}^{\infty}$ is increasing and bounded above by 1. Thus, they are convergent and $\lim_{n\to\infty} a_n = \inf_{n\in N} a_n = a$ and $\lim_{n\to\infty} b_n = \sup_{n\in N} b_n = b$. By the continuity of x_i in $[0,1]$, $x_i(a) = x_i(b) = 0$. We show that $a = 0$ and $b = 1$. If not, namely if $a > 0$ and $b < 1$ then, applying the schema presented above, there exist $0 < \bar{a} < a$ and $1 > \bar{b} > b$, such that $x_i'(\bar{a}) = x_i'(\bar{b}) = 0$, and further $x_i \equiv 0$ in $[\bar{a}, \bar{b}]$, which contradicts definitions of a and b. Therefore $a = 0$ and $b = 1$ and $\bigcup_{n=1}^{\infty}[a_n, b_n] = (0,1)$, so that $x_i \equiv 0$ in $[0,1]$, which contradicts the definition x_i being a nonzero function. Thus (ii) is showed.

(iii) Let $I_i := \inf B_i$ and $S_i := \sup B_i$. If $I_i = 0$ then, we infer the existence of a sequence $\{t_n\} \subset B_i$ such that $t_n \to 0$ as $n \to \infty$, without lost of generality we can assume that $t_1 \geqslant t_n$ for all $n \in N$. Thus for all $n \in N$, $x_i'(t) = 0$ in $[t_n, t_1]$, and further $x_i'(t) = 0$ in $\bigcup_{n=1}^{\infty}[t_n, t_1] = (0, t_1]$, which gives $x_i \equiv const$ in $(0, t_1]$. Taking into account the continuity of x_i in $[0,1]$, we derive that $x_i(t) = 0$ in $(0, t_1]$, which is impossible with respect to the part (ii). The same schema leads to the conclusion that $S_i < 1$.

(iv) By (i) and (iii), we state that $k_i \equiv 0$ in $B_i = [I_i, S_i]$ and $k_i(t) \neq 0$ for all $t \in (0,1) \setminus [I_i, S_i]$. Thus, by the fact that k_i is nonincreasing in $(0,1)$, we infer that $k_i(t) > 0$ for all $t \in (0, I_i)$ and $k_i(t) < 0$ for all $t \in (S_i, 1)$. Finally, taking into account (G3), $x_i'(t) > 0$ for all $t \in (0, I_i)$ and $x_i'(t) < 0$ for all $t \in (S_i, 1)$, which gives the required conclusion. □

2.3. Duality results

In this section we develop the duality theory based on the Fenchel conjugate. To this effect we assume that hypotheses (G1)–(G3) and (S) hold throughout this section. Now we start with the definition of set W^d of arguments of J_D. Let

$$W^d := \big\{\mathbf{p} = (p_1, \dots, p_k) \in W^{1,\mathbf{q}'}\big((0,1), R^k\big)\text{: there exists } \mathbf{x} \in W$$
$$\text{such that } p_i(t) = l_i(t)\big|x_i'(t)\big|^{q_i-2}x_i'(t) \text{ for all } t \in (0,1) \text{ and } i \in \{1, \dots, k\}\big\}.$$

Remark 2. Under conditions (S1)–(S2) we state that for each $\mathbf{x} \in W$ there exists $\mathbf{p} \in W^d$ such that for each $i \in \{1, \dots, k\}$

$$p_i'(\cdot) = -G_i'\big(\cdot, \mathbf{x}(\cdot)\big) \quad \text{a.e. in } (0,1),$$

which means that for almost all $t \in (0,1)$, $-\mathbf{p}'(t) \in \partial G(t, \mathbf{x}(t))$ (∂G is defined by (8)).

Now we perturb J, consider this perturbation (called J_x) on whole space $\mathbf{L}$ and calculate the Fenchel transform of this functional. Since we have no assumptions concerning G outside $(0,1)\times\mathbf{J}$, we consider $\overline{G}$ instead of G, with

$$\overline{G}(t,\mathbf{x}) = \begin{cases} G(t,\mathbf{x}) & \text{if } \mathbf{x}\in\mathbf{I},\ t\in(0,1), \\ +\infty & \text{if } \mathbf{x}\in R^k\setminus\mathbf{I},\ t\in(0,1). \end{cases}$$

So we define the perturbation of J as

$$J_x(\mathbf{g}) = \int_0^1 \left(\overline{G}\big(t,\mathbf{x}(t)+\mathbf{g}(t)\big) - \sum_{i=1}^k \frac{l_i(t)}{q_i}\big|x_i'(t)\big|^{q_i}\right) dt,$$

for all $\mathbf{x} = (x_1,\dots,x_k)\in W$ and $\mathbf{g} = (g_1,\dots,g_k)\in\mathbf{L}$.

Using the fact that we reduce our investigations to the set W which satisfies (S2), we will not change the notation for the action functional containing G or $\overline{G}$.

For $\mathbf{x}\in W$, let us consider a conjugate of $J_x : W^d\to R$ defined by

$$\begin{aligned} J_{\mathbf{x}}^{\#}(\mathbf{p}) &= \sup_{\mathbf{g}\in\mathbf{L}}\left(\int_0^1 \big\langle \mathbf{g}(t)\big|\mathbf{p}'(t)\big\rangle\, dt - J_{\mathbf{x}}(\mathbf{g})\right) \\ &= \sup_{\mathbf{g}\in\mathbf{L}}\left\{\int_0^1 \big\langle \mathbf{g}(t)\big|\mathbf{p}'(t)\big\rangle\, dt - \int_0^1 \overline{G}\big(t,\mathbf{x}(t)+\mathbf{g}(t)\big)\, dt\right\} + \int_0^1 \sum_{i=1}^k \frac{l_i(t)}{q_i}\big|x_i'(t)\big|^{q_i}\, dt. \end{aligned} \tag{11}$$

After simply calculation, we get

$$J_{\mathbf{x}}^{\#}(\mathbf{p}) = -\int_0^1 \big\langle \mathbf{x}(t)\big|\mathbf{p}'(t)\big\rangle\, dt + \int_0^1 \sum_{i=1}^k \frac{l_i(t)}{q_i}\big|x_i'(t)\big|^{q_i}\, dt + \int_0^1 G^*\big(t,\mathbf{p}'(t)\big)\, dt,$$

where G^* is given by $G^*(t,\mathbf{x}^*) := \sup_{\mathbf{x}\in R^k}\{\langle\mathbf{x}^*|\mathbf{x}\rangle - \overline{G}(t,\mathbf{x})\}$ for all $\mathbf{x}^*\in R^k$ and each $t\in(0,1)$ fixed.

As we can see in the proof of the next lemma, Remark 2 allows us to calculate the Fenchel transform of functionals over nonlinear sets W and W^d.

Lemma 4. *For each $\mathbf{x}\in W$,*

$$\inf_{\mathbf{p}\in W^d} J_{\mathbf{x}}^{\#}(-\mathbf{p}) = J(\mathbf{x}). \tag{12}$$

Proof. For a given arbitrary $\mathbf{x}\in W$, we denote by $\bar{\mathbf{p}}_{\mathbf{x}}$ an element of W^d such that $-\bar{\mathbf{p}}_{\mathbf{x}}'(t)\in\partial G(t,\mathbf{x}(t))$ (its existence is due to Remark 2). This fact implies the following chain of inequalities

$$\begin{aligned} \int_0^1 G\big(t,\mathbf{x}(t)\big)\, dt &= \int_0^1 \big\langle -\bar{\mathbf{p}}_{\mathbf{x}}'(t)\big|\mathbf{x}(t)\big\rangle\, dt - \int_0^1 G^*\big(t,-\bar{\mathbf{p}}_{\mathbf{x}}'(t)\big)\, dt \\ &\leqslant \sup_{\mathbf{p}\in W^d}\left\{\int_0^1 \big\langle -\mathbf{p}'(t)\big|\mathbf{x}(t)\big\rangle\, dt - \int_0^1 G^*\big(t,-\mathbf{p}'(t)\big)\, dt\right\} \end{aligned}$$

$$\leqslant \sup_{\mathbf{v}\in\mathbf{L}'}\left(\int_0^1 \langle \mathbf{v}(t)|\mathbf{x}(t)\rangle\,dt - \int_0^1 G^*\big(t,\mathbf{v}(t)\big)\,dt\right)$$

$$= \int_0^1 G^{**}\big(t,\mathbf{x}(t)\big)\,dt = \int_0^1 G\big(t,\mathbf{x}(t)\big)\,dt,$$

where $\mathbf{L}' := L^{q_1'}((0,1),R)\times\cdots\times L^{q_k'}((0,1),R)$, $G^{**}(t,\mathbf{x}^{**}) := \sup_{\mathbf{x}^*\in R^k}\{\langle \mathbf{x}^{**}|\mathbf{x}^*\rangle - G^*(t,\mathbf{x}^*)\}$ for each $t\in(0,1)$ fixed and all $\mathbf{x}^{**}\in R^k$. Due to this chain of inequalities we obtain

$$\begin{aligned}
&\inf_{\mathbf{p}\in W^d} J^{\#}_{\mathbf{x}}(-\mathbf{p})\\
&\quad = -\sup_{\mathbf{p}\in W^d}\left\{\int_0^1 \langle -\mathbf{p}'(t)|\mathbf{x}(t)\rangle\,dt - \int_0^1 G^*\big(t,-\mathbf{p}'(t)\big)\,dt\right\} + \int_0^1 \sum_{i=1}^k \frac{l_i(t)}{q_i}\big|x_i'(t)\big|^{q_i}\,dt\\
&\quad = J(\mathbf{x}). \qquad \square
\end{aligned}$$

Now we define the dual functional $J_D : W^d \to R$ as follows

$$J_D(\mathbf{p}) := \inf_{\mathbf{x}\in W} J^{\#}_{\mathbf{x}}(-\mathbf{p}).$$

It appears that J_D can be rewritten as

$$J_D(\mathbf{p}) = -\sum_{i=1}^k \int_0^1 \frac{1}{q_i'(l_i(t))^{q_i'/q_i}}\big|p_i(t)\big|^{q_i'}\,dt + \int_0^1 G^*\big(t,-\mathbf{p}'(t)\big)\,dt. \tag{13}$$

Indeed, fix $\mathbf{p} = (p_1,\ldots,p_k)\in W^d$. By the definition of W^d there exists $\mathbf{x}_{\mathbf{p}} = (x_{\mathbf{p},1},\ldots,x_{\mathbf{p},k})\in W$ such that for each $i\in\{1,\ldots,k\}$

$$\int_0^1 x'_{\mathbf{p},i}(t)p_i(t)\,dt - \int_0^1 \frac{l_i(t)}{q_i}\big|x'_{\mathbf{p},i}(t)\big|^{q_i}\,dt = \int_0^1 \frac{1}{q_i'(l_i(t))^{q_i'/q_i}}\big|p_i(t)\big|^{q_i'}\,dt.$$

Taking into account the above equality and employing the reasoning presented in the proof of the last lemma we obtain

$$\begin{aligned}
\sum_{i=1}^k\int_0^1 \frac{1}{q_i'(l_i(t))^{q_i'/q_i}}\big|p_i(t)\big|^{q_i'}\,dt &= \sum_{i=1}^k\int_0^1 x'_{\mathbf{p},i}(t)p_i(t)\,dt - \sum_{i=1}^k\int_0^1 \frac{l_i(t)}{q_i}\big|x'_{\mathbf{p},i}(t)\big|^{q_i}\,dt\\
&\leqslant \sup_{\mathbf{x}\in W}\left\{\sum_{i=1}^k\int_0^1\left(x_i'(t)p_i(t) - \frac{l_i(t)}{q_i}\big|x_i'(t)\big|^{q_i}\right)dt\right\}\\
&\leqslant \sum_{i=1}^k\int_0^1 \frac{1}{q_i'(l_i(t))^{q_i'/q_i}}\big|p_i(t)\big|^{q_i'}\,dt
\end{aligned}$$

and further

$$\inf_{\mathbf{x}\in W} J^{\#}_{\mathbf{x}}(-\mathbf{p}) = -\sup_{\mathbf{x}\in W}\left\{\sum_{i=1}^{k}\int_0^1 x_i'(t)p_i(t)\,dt - \sum_{i=1}^{k}\int_0^1 \frac{l_i(t)}{q_i}\left|x_i'(t)\right|^{q_i}dt\right\}$$
$$+\int_0^1 G^*\big(t,-\mathbf{p}'(t)\big)\,dt = -\sum_{i=1}^{k}\int_0^1 \frac{1}{q_i'(l_i(t))^{q_i'/q_i}}\left|p_i(t)\right|^{q_i'}dt + \int_0^1 G^*\big(t,-\mathbf{p}'(t)\big)\,dt,$$

what we have claimed.

Combining Lemma 4 and (13) we infer the following duality result.

Theorem 5.

$$\inf_{\mathbf{x}\in W} J(\mathbf{x}) = \inf_{\mathbf{p}\in W^d} J_D(\mathbf{p}).$$

Our task is now to link the critical points of J with the solutions of (5). Precisely, we shall show the numerical version of this fact and we prove that elements of a minimizing sequence of J approximate (in some sense) the solutions of (5). In addition, we obtain measure of duality gap between the primal and dual functionals for our problem.

Theorem 6. *For each sequence $\{\mathbf{x}_m\}_{m\in N}\subset W$ minimizing $J:W\to R$ there exists minimizing sequence $\{\mathbf{p}_m\}_{m\in N}\subset W^d$ of $J_D:W^d\to R$ satisfying the following relations for each $i\in\{1,\dots,k\}$*

$$-p_{m,i}'(t) = G_i'\big(t,\mathbf{x}_m(t)\big)\quad \text{a.e. in } (0,1) \tag{14}$$

and

$$\lim_{m\to\infty}\int_0^1\left(\frac{1}{q_i'(l_i(t))^{q_i'/q_i}}\left|p_{m,i}(t)\right|^{q_i'} + \frac{l_i(t)}{q_i}\left|x_{m,i}'(t)\right|^{q_i} - x_{m,i}'(t)p_{m,i}(t)\right)dt = 0. \tag{15}$$

Moreover, for a given $\varepsilon>0$ there exists m_0, such that for all $m>m_0$ the following inequality holds

$$\left|J(\mathbf{x}_m) - J_D(\mathbf{p}_m)\right| \leqslant \varepsilon. \tag{16}$$

Proof. We start this proof with the observation that J is bounded below in W, so $-\infty < a := \inf_{\mathbf{x}\in W} J(\mathbf{x}) < +\infty$. Thus for a given $\varepsilon>0$ there exists $m_0\in N$ such that for all $m\geqslant m_0$, $J(\mathbf{x}_m)-a<\varepsilon$. Now, by Remark 2, we can derive that there exists a sequence $\{\mathbf{p}_m\}_{m\in N}\subset W^d$ such that

$$-\mathbf{p}_m'(t)\in\partial G\big(t,\mathbf{x}_m(t)\big)\quad \text{a.e. in } (0,1), \tag{17}$$

for each $m\in N$, so that (14) holds. To show that $\{\mathbf{p}_m\}_{m\in N}$ is a minimizing sequence of J_D, we use the fact that (17) is equivalent to the Fenchel equality for function $\overline{G}$, namely

$$\int_0^1 \overline{G}\big(t,\mathbf{x}_m(t)\big)\,dt = \int_0^1 -G^*\big(t,-\mathbf{p}_m'(t)\big)\,dt - \int_0^1 \big\langle \mathbf{x}_m(t)\big|\mathbf{p}_m'(t)\big\rangle\,dt.$$

Adding $[-\sum_{i=1}^{k}\int_0^1 \frac{l_i(t)}{q_i}|x_i'(t)|^{q_i}\,dt]$ to both sides of the above equality we infer that $J(\mathbf{x}_m) = J^{\#}_{(\mathbf{x}_m)}(-\mathbf{p}_m)$ and further

$$a+\varepsilon > J(\mathbf{x}_m) = J^{\#}_{(\mathbf{x}_m)}(-\mathbf{p}_m) \geqslant \inf_{\mathbf{x}\in W} J^{\#}_{\mathbf{x}}(-\mathbf{p}_m) = J_D(\mathbf{p}_m)$$

for all $m \geqslant m_0$. Combining the last assertion with Theorem 5, we state that $\{\mathbf{p}_m\}_{m\in N}$ is a minimizing sequence of $J_D : W^d \to R$. Taking into account equalities $J_{\mathbf{x}_m}(\mathbf{0}) = -J(\mathbf{x}_m) = -J^{\#}_{\mathbf{x}_m}(-\mathbf{p}_m)$ and $\inf_{m\in N} J_D(\mathbf{p}_m) = \inf_{\mathbf{x}\in W} J(\mathbf{x}) = a$ we state (16) and

$$0 \leqslant J^{\#}_{\mathbf{x}_m}(-\mathbf{p}_m) - J_D(\mathbf{p}_m) \leqslant \varepsilon,$$

which gives (15). □

2.4. The existence of a solution for the BVP

In this section we prove the existence of solution $\bar{\mathbf{x}} \in \overline{W}$ for system (5). As we will show, if we have an explicit definition of W we can usually derive that $\bar{\mathbf{x}}$ is a minimizer of $J : W \to R$. In the other case we can prove that the value of J at $\bar{\mathbf{x}}$ is lower than the infimum of J on W.

Theorem 7. *Under hypotheses* (G1)–(G3) *and* (S) *there exists $\bar{\mathbf{x}} \in \overline{W}$ being a solution of* (5) *such that*

$$\inf_{\mathbf{x}\in W} J(\mathbf{x}) \geqslant J(\bar{\mathbf{x}}). \tag{18}$$

Proof. Our plan is to describe $\bar{\mathbf{x}} \in \overline{W}$ as the weak limit of a certain minimizing sequence (up to subsequence) of $J : W \to R$ and use Theorem 6 to show that $\bar{\mathbf{x}}$ is a solution of (5). Let $S_a = \{x \in W,\ J(\mathbf{x}) \leqslant a\}$ with $a \in R$. It is clear that if a is sufficiently large then S_a is nonempty. Thus we can take a minimizing sequence $\{\mathbf{x}_m\}_{m\in N} = \{(x_{m,1}, \dots, x_{m,k})\}_{m\in N} \subset S_a$, such that each sequence $\{x'_{m,i}\}_{m\in N}$ is bounded in $L^{q_i}((0,1), R)$ and further $\{x_{m,i}\}_{m\in N}$ is bounded in $W_0^{1,q_i}((0,1), R)$. Therefore (going if necessary to a subsequence) for each $i \in \{1, \dots, k\}$, $\{x_{m,i}\}_{m\in N}$ tends weakly to a certain element $\bar{x}_i \in W_0^{1,q_i}((0,1), R)$. This implies uniform convergence of each $\{x_{m,i}\}_{m\in N}$ to $\bar{x}_i$ in $[0,1]$, namely $\mathbf{x}_m \underset{m\to\infty}{\rightrightarrows} \bar{\mathbf{x}}$ in $[0,1]$, where $\bar{\mathbf{x}} := (\bar{x}_1, \dots, \bar{x}_k)$. Moreover $\bar{x}_i \in C([0,1], R)$, $\bar{x}_i(0) = \bar{x}_i(1) = 0$ and $0 \leqslant \bar{x}_i$ on $[0,1]$ for each $i \in \{1, \dots, k\}$.

On the other hand, Theorem 6 guarantees the existence of $\{\mathbf{p}_m\}_{m\in N} \subset W^d$ such that for each $i \in \{1, \dots, k\}$,

$$-p'_{m,i}(t) = G'_i\big(t, \mathbf{x}_m(t)\big) \quad \text{a.e. in } (0,1), \tag{19}$$

$$0 = \lim_{m\to\infty} \int_0^1 \left(\frac{1}{q'_i (l_i(t))^{q'_i/q_i}} \left|p_{m,i}(t)\right|^{q'_i} + \frac{l_i(t)}{q_i} \left|x'_{m,i}(t)\right|^{q_i} + x_{m,i}(t) p'_{m,i}(t) \right) dt. \tag{20}$$

Due to (19) and (G2), $\{p'_{m,i}\}_{m\in N}$ is bounded in $L^{q'_i}((0,1), R)$ so that there exists $z_i \in L^{q'_i}((0,1), R)$ such that $p'_{m,i} \rightharpoonup z_i$ (up to a subsequence) in $L^{q'_i}((0,1), R)$. Taking into account the above reasoning, (20) and the boundedness of $\{x'_{m,i}\}_{m\in N}$ and $\{x_{m,i}\}_{m\in N}$ in $L^{q_i}((0,1), R)$ and $C([0,1], R)$, respectively, we state the boundedness of $\{p_{m,i}\}_{m\in N}$ in $L^{q'_i}((0,1), R)$. In consequence, up to a subsequence, $\{p_{m,i}\}_{m\in N}$ is weakly convergent in $L^{q'_i}((0,1), R)$. Finally, $\{p_{m,i}\}_{m\in N}$ is weakly convergent in $W^{1,q'_i}((0,1), R)$ to $\bar{p}_i \in W^{1,q'_i}((0,1), R)$ and further $\{p_{m,i}\}_{m\in N}$ is uniformly convergent to $\bar{p}_i$ and $\bar{p}_i \in C((0,1), R)$.

Applying again (19) and (20) we get for all $i \in \{1, \dots, k\}$

$$\begin{aligned} 0 &= \lim_{m\to\infty} \int_0^1 \left(\frac{1}{q'_i (l_i(t))^{q'_i/q_i}} \left|p_{m,i}(t)\right|^{q'_i} + \frac{l_i(t)}{q_i} \left|x'_{m,i}(t)\right|^{q_i} - x'_{m,i}(t) p_{m,i}(t) \right) dt \\ &\geqslant \int_0^1 \frac{1}{q_i (l_i(t))^{q'_i/q_i}} \left|\bar{p}_i(t)\right|^{q'_i} dt + \int_0^1 \frac{l_i(t)}{q_i} \left|\bar{x}'_i(t)\right|^{q_i} dt - \int_0^1 \bar{x}'_i(t) \bar{p}_i(t)\, dt \end{aligned}$$

and

$$0 = \liminf_{m\to\infty} \int_0^1 \left[G^*\left(t, -p'_{m,i}(t)\right) + G\left(t, x_{m,i}(t)\right) + x_{m,i}(t)p'_{m,i}(t)\right] dt$$
$$\geqslant \int_0^1 \left[G^*\left(t, -\bar{p}'_i(t)\right) + G\left(t, \bar{x}_i(t)\right) + \bar{x}_i(t)\bar{p}'_i(t)\right] dt.$$

Therefore, by the properties of the Fenchel transform, we get for each $i \in \{1, \dots, k\}$

$$\begin{aligned} \bar{p}'_i(t) &= -G'_i\left(t, \bar{\mathbf{x}}(t)\right) \quad \text{a.e. in } (0,1), \\ \bar{p}_i(t) &= l_i(t)\left|\bar{x}'_i(t)\right|^{q_i-2}\bar{x}'_i(t) \quad \text{on } (0,1). \end{aligned} \tag{21}$$

Substituting the last assertion into (21), we obtain $l_i|\bar{x}'_i|^{q_i-2}\bar{x}'_i \in W^{1,q'_i}((0,1),R)$ and state that $\bar{\mathbf{x}} \in \overline{W}$ is a solution of (5). (18) is a simply consequence of weak lower semicontinuity of J in A_0^k and thus also in $\overline{W}$. $\square$

Corollary 8. *Suppose that* (G1)–(G4) (*and* (G5)) *hold with* $q_i = 2$, *for* $i = 1, \dots, k$. *Then*

(i) *there exists a solution* $\bar{\mathbf{x}} \in W_1$ ($\bar{\mathbf{x}} \in W_2$) *for* (5) *such that*

$$\inf_{\mathbf{x}\in W_1} J(\mathbf{x}) = J(\bar{\mathbf{x}})\left(\inf_{\mathbf{x}\in W_2} J(\mathbf{x}) = J(\bar{\mathbf{x}})\right);$$

(ii) *if we assume additionally that* $G'_i \in C((0,1)\times\mathbf{J})$ *we state that* $\bar{\mathbf{x}}$ *is also a classical solution of* (5).

3. Continuous dependence on functional parameters

Let us consider the case when nonlinearities in system (5) depend on functional parameter $\mathbf{z} : (0,1) \to \mathbf{D} \subset R^s$, $\mathbf{z} \in \mathbf{Z}$, where $\mathbf{Z}$ is a certain subset of $L^p(0,1)$, $s \in \mathbf{N}$, $p > 1$, namely we come back to system (4). It is clear that we have to make conditions on the nonlinearities which guarantee that for each $\mathbf{z} \in \mathbf{Z}$ function $(0,1)\times\mathbf{J} \ni (t,\mathbf{x}) \mapsto \widetilde{G}(t,\mathbf{x},\mathbf{z}(t))$ satisfies hypotheses (G1)–(G3) and there exists $W_z \subset \overline{W}$ satisfying (S1)–(S2). Thus, we assume that

(G1z) there exist k-dimensional interval $\mathbf{I} := [0,\tilde{d}_1]\times\cdots\times[0,\tilde{d}_k]$, $\tilde{d}_1,\dots,\tilde{d}_k \in R_+$, and a neighborhood $\mathbf{J}$ of $\mathbf{I}$ such that $\widetilde{G} \in C((0,1)\times\mathbf{J}\times\mathbf{D}, R)$ and $\widetilde{G}$ is convex with respect to the second variable in $\mathbf{J}$ for each $\mathbf{z} \in \mathbf{Z}$ and $t \in (0,1)$, $|\int_0^1 \widetilde{G}(t,\mathbf{0},\mathbf{z}(t))\,dt| < \infty$;

(G2z) for each $i \in \{1,\dots,k\}$ partial derivative $\widetilde{G}'_i = \frac{\partial\widetilde{G}}{\partial x_i}$ exists and is nonnegative in $(0,1)\times\mathbf{J}\times\mathbf{D}$, $\int_0^1 \widetilde{G}'_i(t,\mathbf{0},\mathbf{z}(t))\,dt \neq 0$ and there exits $\varphi_i \in L^{q'_i}((0,1),R)$ such that

$$\sup_{\mathbf{x}\in\mathbf{I}} \widetilde{G}'_i\left(t,\mathbf{x},\mathbf{z}(t)\right) \leqslant \varphi_i(t) \quad \text{a.e. in } (0,1) \tag{22}$$

for all $i \in \{1,\dots,k\}$ and all $\mathbf{z} \in \mathbf{Z}$;

(Sz) for each $\mathbf{z} \in \mathbf{Z}$, there exists a nonempty set $W_z \subset \overline{W}$ satisfying the following conditions:

(S1z) for each $\mathbf{x} \in W_\mathbf{z}$ there exists $\tilde{\mathbf{x}} \in W_\mathbf{z}$ such that for all $i \in \{1,\dots,k\}$

$$-\left(l_i(t)\left|\tilde{x}'_i(t)\right|^{q_i-2}\tilde{x}'_i(t)\right)' = G'_i\left(t,\mathbf{x}(t),\mathbf{z}(t)\right) \quad \text{a.e. in } (0,1),$$
$$\tilde{x}_i(0) = \tilde{x}_i(1) = 0,$$

(S2) for each $\mathbf{x} \in W_\mathbf{z}$, $\mathbf{x}(t) \in \mathbf{I}$ for all $t \in [0,1]$.

Theorem 9. *Suppose that* (G1z), (G2z), (G3), (Sz) *hold and the sequence of parameters* $\{\mathbf{z}_m\}_{m=1}^{\infty} \subset \mathbf{Z}$ *is a.e. convergent to* $\mathbf{z}_0 \in \mathbf{Z}$ *in* $(0,1)$. *Let us denote by* $\bar{\mathbf{x}}_m \in W_{z_m}$ *a solution of* (4) *with* $\mathbf{z} = \mathbf{z}_m$, $m \in \mathbf{N}$. *Then there exists a subsequence, still denoted by* $\{\bar{\mathbf{x}}_m\}_{m=1}^{\infty}$, *uniformly convergent to* $\bar{\mathbf{x}}_0 \in \overline{W}$ *which is a solution of* (4) *with* $\mathbf{z} = \mathbf{z}_0$ *and such that* $\bar{\mathbf{x}}_0(t) \in \mathbf{I}$ *for all* $t \in [0,1]$.

Proof. By Theorem 7 for each $m \in \mathbf{N}$, there exists solution $\bar{\mathbf{x}}_m \in W_{z_m} \subset \overline{W}$ (dependent on $\mathbf{z}_m$). We start our proof with the observation that the definition of W_{z_m} implies the boundedness of $\{\bar{\mathbf{x}}_m\}_{m=1}^{\infty}$ in $C([0,1],R^k)$. Since $\bar{\mathbf{x}}_m \in W_{z_m}$ is a solution of (4) with $\mathbf{z} = \mathbf{z}_m$, for all $m \in \mathbf{N}$, we have for each $i \in \{1,\dots,k\}$

$$\int_0^1 \left|\bar{x}'_{m,i}(t)\right|^{q_i} dt \leqslant \frac{1}{l_{\min}} \int_0^1 \left(-l_i(t)\left|\bar{x}'_{m,i}(t)\right|^{q_i-2}\bar{x}'_{m,i}(t)\right)'\bar{x}_{m,i}(t)\,dt$$

$$= \frac{1}{l_{\min}} \int_0^1 \widetilde{G}'_i\big(t,\bar{\mathbf{x}}_m(t),\mathbf{z}_m(\mathbf{t})\big)\bar{x}_{m,i}(t)\,dt \leqslant \frac{\tilde{d}_i}{l_{\min}} \int_0^1 \varphi_i(y)\,dt,$$

so that $\{\bar{x}'_{m,i}\}_{m=1}^{\infty}$ is bounded in $L^{q_i}((0,1),R)$. Therefore for each $i \in \{1,\dots,k\}$, $\{\bar{x}_{m,i}\}_{m=1}^{\infty}$ (up to a subsequence) is weakly convergent in $W_0^{1,q_i}((0,1),R)$ to a certain $\bar{x}_{0,i} \in W_0^{1,q_i}((0,1),R)$ and further uniformly convergent to $\bar{x}_{0,i}$ in $[0,1]$. Consequently $\bar{x}_{0,i} \in C([0,1],R)$. This implies that for all $i \in \{1,\dots,k\}$, $\bar{x}_{0,i}(t) \in \mathbf{I}$ for $t \in [0,1]$. Now we consider sequence $\{\mathbf{p}_m\}_{m\in N} \subset \mathbf{L}'$, given by $\mathbf{p}_m := (p_{m,1},\dots,p_{m,k}) \subset \mathbf{L}'$, where

$$p_{m,i}(t) := l_i(t)\left|\bar{x}'_{m,i}(t)\right|^{q_i-2}\bar{x}'_{m,i}(t) \quad \text{in } (0,1) \tag{23}$$

for each $i \in \{1,\dots,k\}$. Since for each $m \in \mathbf{N}$, $\bar{\mathbf{x}}_m$ is the solution of (4) corresponding to parameter $\mathbf{z}_m$ we get

$$p'_{m,i}(t) = -\widetilde{G}'_i\big(t,\bar{\mathbf{x}}_m(t),\mathbf{z}_m(\mathbf{t})\big) \quad \text{a.e. in } (0,1). \tag{24}$$

Combining both assertions ((23) and (24)) with condition (G2z)-(22) and the boundedness of $\{\bar{x}'_{m,i}\}_{m=1}^{\infty}$ in $L^{q_i}((0,1),R)$, we derive the existence of a subsequence still denoted by $\{p_{m,i}\}_{m\in N}$ weakly convergent to a certain $p_{0,i}$ in $W^{1,q'_i}((0,1),R)$. Thus $\{p_{m,i}\}_{m\in N}$ is uniformly convergent to $p_{0,i}$. Thus $p_{0,i} \in C((0,1),R)$. As in the proof of Theorem 9 we obtain for each $i \in \{1,\dots,k\}$

$$0 = \liminf_{m\to\infty} \int_0^1 \left(\frac{l_i(t)}{q_i}\left|\bar{x}'_{m,i}(t)\right|^{q_i} + \frac{1}{q'_i(l_i(t))^{q'_i/q_i}}\left|p_{m,i}(t)\right|^{q'_i} + \bar{x}'_{m,i}(t)p_{m,i}(t)\right)$$

$$\geqslant \int_0^1 \left(\frac{l_i(t)}{q_i}\left|\bar{x}'_{0,i}(t)\right|^{q_i} + \frac{1}{q'_i(l_i(t))^{q'_i/q_i}}\left|p_{0,i}(t)\right|^{q'_i} + \bar{x}'_{0,i}(t)p_{0,i}(t)\right) dt$$

and

$$0 = \liminf_{m\to\infty} \int_0^1 \left[\widetilde{G}^*\big(t,-p'_{m,i}(t),z_m(t)\big) + \widetilde{G}\big(t,x_{m,i}(t),z_m(t)\big) + x_{m,i}(t)p'_{m,i}(t)\right] dt$$

$$\geqslant \int_0^1 \left[\widetilde{G}^*\big(t,-p'_{0,i}(t),z_0(t)\big) + \widetilde{G}\big(t,\bar{x}_{0,i}(t),z_0(t)\big) + \bar{x}_{0,i}(t)p'_{0,i}(t)\right] dt,$$

where $\widetilde{G}^*(t,x^*,z) := \sup_{x\in R}\{xx^* - \widetilde{G}(t,x,z)\}$ for all $(t,x^*,z) \in (0,1)\times R\times \mathbf{D}$. Finally, by the proprieties of the Fenchel transform, we derive for each $i \in \{1,\dots,k\}$

$$p'_{0,i}(t) = -\widetilde{G}'_i\big(t, \bar{\mathbf{x}}_0(t), \mathbf{z}_0(\mathbf{t})\big) \quad \text{a.e. in } (0,1) \tag{25}$$

and

$$p_{0,i}(t) := l_i(t)\left|\bar{x}'_{0,i}(t)\right|^{q_i-2}\bar{x}'_{0,i}(t) \quad \text{in } (0,1).$$

Substituting the last assertion into (25) we state that $\bar{\mathbf{x}}_0 \in \overline{W}$ is a positive solution of (4) with $\mathbf{z} = \mathbf{z}_0$. $\square$

Remark 3. Theorem 9 covers the case when values of nonlinearities for parameters $\mathbf{z}$ in (4) are small enough in some sense – see assumption (G2z)-(22).

Remark 4. Let us consider the case when $q_i = 2$ for all $i \in \{1,\dots,k\}$ and

(G2z′) there exists positive number d such that $d \leqslant \min_{i\in\{1,\dots,k\}} \tilde{d}_i$ and $\int_0^1 \varphi_i(t)\,dt \leqslant 4dc_i l^2_{\min}$, where φ is given in (G2z).

Then Lemma 1 leads to the conclusion that for each $z \in Z$, W_1 satisfies assumption (Sz). Thus Theorem 9 is still valid if we assume (G1z)–(G2z), (G2z′), (G3), omitting (Sz).

As an immediate consequence of Theorem 9 and the above remark we get the following corollary.

Corollary 10. *Assume* (G1z), (G2z), (G2z′) *and* (G3) *with* $q_i = 2$ *and consider a sequence of parameters* $\{\mathbf{z}_m\}_{m=1}^{\infty} \subset \mathbf{Z}$*, which is a.e. convergent to* $\mathbf{z}_0 \in \mathbf{Z}$ *in* $(0,1)$*. Let us denote by* $\bar{\mathbf{x}}_m \in W_1$ *a solution of* (4) *with* $\mathbf{z} = \mathbf{z}_m$*,* $m \in \mathbf{N}$*. Then there exists a subsequence, still denoted by* $\{\bar{\mathbf{x}}_m\}_{m=1}^{\infty}$*, uniformly convergent to* $\bar{\mathbf{x}}_0 \in W_1$ *which is a solution of* (4) *with* $\mathbf{z} = \mathbf{z}_0$*. If, in addition,* $\widetilde{G}'_i \in C((0,1)\times \mathbf{J}\times \mathbf{R}^s, R)$*, for all* $i = 1,\dots,k$*,* $\mathbf{Z} \subset C((0,1),R^k) \cap L^2((0,1),R^k)$ *then* $\bar{\mathbf{x}}_0 \in W_1 \cap C^2((0,1),R^k)$*, where* W_1 *is given in Lemma* 1.

4. Nonexistence result for positive solutions

It is clear that assumption (G2z) (together with (G1z)) guarantees the existence of at least one positive solution of (4) for each $\mathbf{z} \in \mathbf{Z}$. Moreover we derived that these solutions are bounded by a common, pre-specified constant independent of $\mathbf{z}$. Now we would like to ask about an a priori bounds of solutions (if they exist) for (4) when we omit (22) in (G2z) and $\widetilde{G}$ satisfies (G1z)–(G2z) for $\mathbf{J} := [0,+\infty)$. We want to obtain results analogous to the results presented in [20], where the authors consider (4) in special form. As in [20], we employ the schema based on the fact that a solution of (4) is a fixed point of a certain operator $\mathbf{A}$. We consider (4) with $q_i = 2$ and use $\mathbf{A}$ defined in Section 2.1. We start with the case when $\widetilde{G}'_i$ is superlinear at infinity uniformly in $\mathbf{z} \in \mathbf{Z}$.

Theorem 11. *Let us consider the case when* $q_i = 2$ *and assume* (G1z)–(G2z) *for* $\mathbf{J} := [0,+\infty)^k$ *without* (22) *and, additionally*

(G6z) *there exist* $i_0 \in \{1,\dots,k\}$ *and a subset* $B \subset (0,1)$ *of positive measure such that*

$$\lim_{\|\mathbf{u}\|\to+\infty} \frac{\inf_{\mathbf{z}\in\mathbf{Z}} \widetilde{G}'_{i_0}(t,\mathbf{u},\mathbf{z})}{\|\mathbf{u}\|} = +\infty \quad \text{uniformly w.r.t. } t \in B.$$

Then solutions of (4) *(if they exist) are bounded in* $\|\cdot\|_\infty$ *norm by a common constant* $M_1 > 0$ *independent of* $\mathbf{z}$*.*

Proof. Assume otherwise, that there exists a sequence $\{\mathbf{x}_m\}_{m=1}^{\infty}$ of solutions for (4) such that $\|\mathbf{x}_m\|_\infty \to +\infty$ as $m \to +\infty$. Then we have

$$\|\mathbf{x}_m\|_\infty \geqslant x_{m,i_0}(t_0) = A_{i_0}\mathbf{x}_m(t_0) \geqslant \int_0^1 \mathbf{G}_{i_0}(s,t_0) \inf_{\mathbf{z}\in\mathbf{Z}} \widetilde{G}'_{i_0}\big(s,\mathbf{x}_m(\mathbf{s}),\mathbf{z}(s)\big)\,ds$$

$$\geqslant \|\mathbf{x}_m\|_\infty \int_B \mathbf{G}_{i_0}(s,t_0) \frac{\inf_{\mathbf{z}\in\mathbf{Z}} \widetilde{G}'_{i_0}(s,\mathbf{x}_m(s),\mathbf{z}(s))}{\|x_m\|_\infty}\,ds,$$

for all $m \in N$ and certain $t_0 \in (0,1)$, and further

$$\int_B \mathbf{G}_{i_0}(s,t_0) \frac{\inf_{\mathbf{z}\in\mathbf{Z}} \widetilde{G}'_{i_0}(s,\mathbf{x}_m(s),\mathbf{z}(s))}{\|x_m\|_\infty}\,ds \leqslant 1, \quad \text{for all } m \in N,$$

which contradicts condition (G6z). □

We can prove analogous result in the sublinear case.

Theorem 12. *Let us consider the case when $q_i = 2$ and assume* (G1z)–(G2z) *for* $\mathbf{J} := [0,+\infty)$ *without* (22) *and, additionally*

(G7z) *for all* $i \in \{1,\dots,k\}$, $\lim_{\|\mathbf{u}\|\to+\infty} \frac{\sup_{\mathbf{z}\in\mathbf{Z}} \widetilde{G}'_i(t,\mathbf{u},\mathbf{z})}{\|\mathbf{u}\|} = 0$ *uniformly w.r.t.* $t \in [0,1]$,

then the solutions of (4) (*if exist*) *are bounded in* $\|\cdot\|_\infty$ *norm by a common constant* $M_2 > 0$ *independent of* $\mathbf{z}$.

Proof. As in the proof of the previous theorem, we suppose the existence of a sequence $\{\mathbf{x}_m\}_{m=1}^{\infty}$ of solutions of (4) such that $\|\mathbf{x}_m\|_\infty \to +\infty$ as $m \to +\infty$. Thus, for all $t \in (0,1)$

$$\big\|\mathbf{x}_m(t)\big\|^2 = \sum_{i=1}^k \big\|A_i\mathbf{x}_m(t)\big\|^2 \leqslant \sum_{i=1}^k \left(\int_0^1 \mathbf{G}_i(s,t) \sup_{\mathbf{z}\in\mathbf{Z}} \widetilde{G}'_i\big(s,\mathbf{x}_m(\mathbf{s}),\mathbf{z}(s)\big)\,ds\right)^2$$

$$\leqslant \frac{l_{\max}^2}{l_{\min}^4} \frac{\|\mathbf{x}_m\|_\infty^2}{16} \sum_{i=1}^k \left(\int_0^1 \sup_{\mathbf{z}\in\mathbf{Z}} \widetilde{G}'_i\big(s,\mathbf{x}_m(s),\mathbf{z}(s)\big)/\|x_m\|_\infty\,ds\right)^2,$$

and further

$$\sum_{i=1}^k \left(\int_0^1 \sup_{\mathbf{z}\in\mathbf{Z}} \widetilde{G}'_i\big(s,\mathbf{x}_m(s),\mathbf{z}(s)\big)/\|x_m\|_\infty\,ds\right)^2 \geqslant 16\frac{l_{\min}^4}{l_{\max}^2},$$

which contradicts condition (G7z). □

As a consequence of the previous theorem we obtain nonexistence result.

Theorem 13. *Let us consider the case when $q_i = 2$. Suppose that* (G1z)–(G2z) *without* (22), (G3), (G6z) (*or* (G7z)) *hold, and assume additionally*

(G8z) *there exists a sequence of parameters* $\{\mathbf{z}_m\}_{m\in N} \subset \mathbf{Z}$ *such that for certain* $i_0 \in \{1,\dots,k\}$, $\inf_{\mathbf{u}\in\mathbf{I}} \widetilde{G}'_{i_0}(t,\mathbf{u},\mathbf{z}_m(t)) \to \infty$ *if* $m\to\infty$ *(uniformly w.r.t.* $t\in[0,1]$*).*

Then there exists $m_0 \in N$ *such that* (4) *does not possess any positive solutions for parameters* $\mathbf{z}_m$ *with* $m > m_0$.

Proof. If we suppose otherwise, then for all $m \in N$ there would exist a solution x_m of (4) with parameter $\mathbf{z}_m$, where $\{\mathbf{z}_m\}_{m\in N} \subset \mathbf{Z}$ satisfies (G8z). But then for m sufficiently large

$$\inf_{u\in\mathbf{I}} \widetilde{G}'_{i_0}\big(t,\mathbf{u},\mathbf{z}_m(t)\big) \geqslant \big(\max\{M_1,M_2\}+1\big)\left(\int_0^1 \mathbf{G}_{i_0}(s,t_0)\,ds\right)^{-1}$$

for M_1, M_2 given in previous theorems and fixed $t_0 \in (0,1)$. Hence

$$\begin{aligned}\max\{M_1,M_2\} &\geqslant \|\mathbf{x}_m\|_\infty \geqslant x_{m,i}(t_0) = A_{i_0}\mathbf{x}_m(t_0)\\ &\geqslant \int_0^1 \mathbf{G}_{i_0}(s,t_0)\inf_{\mathbf{u}\in\mathbf{I}} \widetilde{G}'_{i_0}\big(s,\mathbf{u}(\mathbf{s}),\mathbf{z}_m(s)\big)\,ds\\ &\geqslant \max\{M_1,M_2\}+1,\end{aligned}$$

which is impossible. □

Remark 5. Let us note that (G8z) implies nonexistence of positive solutions of (4) with parameters, for which nonlinearities are too large. On the other hand (22) in (G2z) guarantees that for all $\mathbf{z}\in\mathbf{Z}$, nonlinearities are small enough (in some sense) to obtain the solvability of (4).

5. Radial solutions of the system of PDEs

5.1. Radial solutions in exterior domains

In this section we will present the application of the above results to the systems of PDEs of elliptic type similar to (2) or more general one given by

$$-\operatorname{div}\big(\dot{l}_i\big(\|y\|\big)\nabla v_i\big) = F'_i\big(\|y\|,\mathbf{v}\big) \quad \text{for } y\in\Omega, \tag{26}$$

$$v_i(y) = 0 \quad \text{for } \|y\| = 1, \tag{27}$$

$$\lim_{\|y\|\to\infty} v_i(y) = 0, \tag{28}$$

for all $i\in\{1,\dots,k\}$, where $\mathbf{v} := (v_1,\dots,v_k)$, $\Omega = \{y\in R^n,\ \|y\|>1\}$, $n\in\mathbf{N}$, $n>2$. As we mentioned in Section 1, investigation of the existence of radial solutions for (26) leads to the investigation of the singular Dirichlet problem (5) with nonlinearities G'_i given by

$$\begin{aligned}G'_i(t,\mathbf{v}) &= \frac{1}{(n-2)^2}(1-t)^{\frac{2n-2}{2-n}} F'_i\big((1-t)^{\frac{1}{2-n}},\mathbf{v}\big),\\ l_i(t) &= \dot{l}_i\big((1-t)^{\frac{1}{2-n}}\big).\end{aligned}$$

It is due to the fact that if $\mathbf{v}(y) = (a_i(\|y\|))_{i=1}^k$ in $y \in \Omega$, with $a_i : [1, +\infty) \to R$ for all $i \in \{1, \ldots, k\}$, is a solution of (26)–(28), then $\mathbf{x}(t) = (a_i((1-t)^{\frac{1}{2-n}}))_{i=1}^k$ in $(0,1)$, satisfies (5) and otherwise, having a solution $\mathbf{x}$ of system (5) we can state that for $\mathbf{v}(y) := (x_i(1-\|y\|^{2-n}))_{i=1}^k$ in Ω, (26)–(28) holds.

We start our consideration with the assumptions which guarantee hypotheses (G1)–(G4) for the nonlinearities G_i'

(F1) there exist k-dimensional interval $\mathbf{I} := [0, \tilde{d}_1] \times \cdots \times [0, \tilde{d}_k]$, $\tilde{d}_1, \ldots, \tilde{d}_k \in (0, +\infty)$, and a neighborhood $\mathbf{J}$ of $\mathbf{I}$ such that function $F : (1, +\infty) \times \mathbf{J} \to (0, +\infty)$, $F \in C((1, +\infty) \times \mathbf{J}, R)$ and F is convex with respect to the second variable in $\mathbf{J}$, $|\int_1^\infty l^{n-1} F(l, \mathbf{0})\, dl| < \infty$;

(F2) for each $i \in \{1, \ldots, k\}$, the partial derivative $F_i' := \frac{\partial F}{\partial v_i}$ exists, is continuous and nonnegative in $(1, +\infty) \times \mathbf{J}$, function $l \mapsto l^{2n-2+\frac{1-n}{q'}} \sup_{\mathbf{v} \in \mathbf{I}} F_i'(l, \mathbf{v})$ belongs to $L^{q'}((1, +\infty), R)$,

$$\int_1^\infty l^{n-1} F_i'(l, \mathbf{0})\, dl \neq 0;$$

(F3) for each $i \in \{1, \ldots, k\}$, $\dot{l}_i \in C([1, +\infty), R) \cap C^1((1, +\infty), R)$ and there exist $l_{\min}, l_{\max} > 0$ such that for all $t \in [1, +\infty)$, $l_{\min} \leqslant \dot{l}_i(t) \leqslant l_{\max}$;

(F4) there exists $0 < d \leqslant \min_{i \in \{1,\ldots,k\}} \tilde{d}_i$ such that for all $i \in \{1, \ldots, k\}$

$$\frac{1}{(n-2)} \int_1^\infty l^{n-1} \sup_{\mathbf{v} \in \mathbf{I}} F_i'(l, \mathbf{v})\, dl \leqslant 4 d c_i l_{\min}^2.$$

An immediate consequence of Corollary 8 is the result concerning the radial solutions for elliptic BVP (26)–(28).

Theorem 14. *Let us assume that F satisfies hypotheses* (F1)–(F4)*. Then* (26)–(28) *possesses at least one positive radial classical solution $\bar{\mathbf{v}} = (\bar{v}_1, \ldots, \bar{v}_k)$ such that for all $i \in \{1, \ldots, k\}$, $\bar{v}_i(y) \leqslant d$ for $y \in \Omega$.*

Let us note that Section 2.2 allows us to derive some properties of radial solutions of (26)–(28).

Proposition 15. *Assume that* (F1)–(F4) *hold. If $\mathbf{u}$ is a radial, nonnegative solution of* (26)–(28) *such that $\mathbf{u}(y) := (u_1(y), \ldots, u_k(y)) \in \mathbf{I}$ for $y \in \Omega$, then for each $i \in \{1, \ldots, k\}$*

(i) *there exist $1 < r_i < R_i$ such that $\widetilde{B}_i := \{y \in \Omega,\ \nabla u_i(y) = 0\} = \{y \in \Omega,\ r_i \leqslant \|y\| \leqslant R_i\}$;*
(ii) *$u_i(y) > 0$ for all $y \in \Omega$;*
(iii) *u_i is radially strictly increasing in $\Omega_1 := \{y \in \Omega,\ \|y\| \leqslant r_i\}$, namely for all $y_1, y_2 \in \Omega_1$*

$$\|y_1\| < \|y_2\| \quad \Rightarrow \quad u_i(y_1) < u_i(y_2)$$

and u_i is radially strictly decreasing in $\Omega_2 := \{y \in \Omega,\ \|y\| \geqslant R_i\}$, namely for all $y_1, y_2 \in \Omega_2$

$$\|y_1\| < \|y_2\| \quad \Rightarrow \quad u_i(y_1) > u_i(y_2);$$

(iv) *for each $y_0 \in \widetilde{B}_i$, $u_i(y_0) = \max_{y \in \Omega} u_i(y)$.*

Proof. By the reasoning presented at the beginning of this section, we state that

$$\mathbf{u}(y) = \mathbf{x}\big(1 - \|y\|^{2-n}\big) \quad \text{in } y \in \Omega, \tag{29}$$

where $\mathbf{x} : [0,1] \to [0, +\infty)$ is a solution of (5) with $q_i = 2$. Since $\mathbf{u} \not\equiv 0$ in Ω (by (F2)) and $\mathbf{u}(y) \in \mathbf{I}$ in Ω, we have $\mathbf{x} \not\equiv 0$ and $\mathbf{x}(t) \in \mathbf{I}$ in $(0,1)$. Fix $i \in \{1, \ldots, k\}$.

(i) We start with the observation that for $y \in \Omega$, $\nabla u_i(y) = 0$ if and only if $x_i'(1 - \|y\|^{2-n}) = 0$. Thus, by the first part of Proposition 3

$$I_i \leqslant 1 - \|y\|^{2-n} \leqslant S_i,$$

which is equivalent to the chain of inequalities

$$(1 - I_i)^{\frac{1}{2-n}} \leqslant \|y\| \leqslant (1 - S_i)^{\frac{1}{2-n}},$$

where $I_i := \inf\{t \in (0,1), x_i'(t) = 0\}$ and $S_i := \sup\{t \in (0,1), x_i'(t) = 0\}$. So that $\widetilde{B}_i = \{y \in \Omega, r_i \leqslant \|y\| \leqslant R_i\}$ with $r_i := (1 - I_i)^{\frac{1}{2-n}}$ and $R_i := (1 - S_i)^{\frac{1}{2-n}}$.

(ii) Assertion (29) and the second part of Proposition 3 lead to the require conclusion.

(iii) To prove this property of $\mathbf{u}$ we take $y_1, y_2 \in \Omega_1$ such that $\|y_1\| < \|y_2\|$. Let $t_1 := 1 - \|y_1\|^{2-n}$ and $t_2 := 1 - \|y_2\|^{2-n}$. Then $0 < t_1 < t_2 \leqslant I_i$, so, by the fourth part of Proposition 3, we infer the relation

$$u_i(y_1) = x_i\big(1 - \|y_1\|^{2-n}\big) < x_i\big(1 - \|y_2\|^{2-n}\big) = u(y_2).$$

In the same way we show the other assertion of (iii).

(iv) This equality is a simply consequence of parts (i) and (iii). $\square$

Now we give examples of superlinear and sublinear problems (26)–(28) consisting of two elliptic equations with nonlinearities satisfying (F1)–(F4) with exterior domain $\Omega \subset R^3$.

Example 1. We study the following problem

$$\begin{cases} -\operatorname{div}\big(\big(\|y\|^{-2} + 1\big)\nabla u(y)\big) = \|y\|^{-4}\Big(\big(u(y) - w(y) + 2\big) + \dfrac{1}{8}\|y\|^{-2}\big(u(y)\big)^3\Big), \\ -\operatorname{div}\big(\big(\|y\|^{-3} + 1\big)\nabla w(y)\big) = \|y\|^{-4}\Big(\big(u(y) - w(y) + 3\big) + \dfrac{3}{128}\|y\|^{-2}\big(w(y)\big)^5\Big) \end{cases}$$

for $y \in \Omega := \{y \in R^3, \|y\| > 1\}$ with boundary conditions (27)–(28).

We can state that in this case function

$$F(y,u,w) = \frac{1}{2}\|y\|_{R^3}^{-4}\left((u-w)^2 + 4u + 6w + \|y\|^{-2}\left(\left(\frac{u}{2}\right)^4 + \frac{1}{2}\left(\frac{w}{2}\right)^6\right)\right)$$

satisfies conditions (F1)–(F4) for $\mathbf{J} := (-1,3) \times (-1,3)$ and $d = 2$ ($c_1 = \int_0^1 \frac{dt}{(1-t)^2+1} \approx 0.785$, $c_2 = \int_0^1 \frac{dt}{(1-t)^3+1} \approx 0.836$, $l_{\min} = 1$). Thus, by Theorem 14, there exists a positive radial solution $(\bar{u}, \overline{w})$ such that $\bar{u}(y) \leqslant 2$ and $\overline{w}(y) \leqslant 2$ for $y \in \Omega$.

Example 2. Let us consider the system

$$\begin{cases} -\operatorname{div}\big(\big(\|y\|^{-2} + 1\big)\nabla u(y)\big) = \|y\|^{-4}\Big(\big(u(y) - w(y)\big) + e^{u(y)}\Big(1 + \dfrac{1}{2}\|y\|^{-4}\Big)\Big), \\ -\operatorname{div}\big(\big(\|y\|^{-1} + 1\big)\nabla w(y)\big) = \|y\|^{-4}\Big(\big(u(y) - w(y)\big) + \dfrac{(5 + w(y))}{(3 - w(y)) * (4 - w(y))}\Big) \end{cases}$$

for $y \in \Omega := \{y \in R^3, \|y\| > 1\}$ with boundary conditions (27)–(28). We can state that F given by

$$F\big(\|y\|_{R^n}, u, w\big) = \frac{1}{2}\big(\|y\|\big)^{-4}\bigg((u-w)^2 + \bigg(1 + \frac{1}{2}\|y\|^{-4}\bigg)e^u - 4\ln|3-w| + 5\ln|4-w|\bigg)$$

satisfies (F1)–(F4), for $\mathbf{J} := (-1, \frac{5}{2}) \times (-1, \frac{5}{2})$ and $d = 2$ ($c_1 = \frac{\pi}{4}$, $c_2 = \ln 2$, $l_{\min} = 1$). By Theorem 14, our problem possesses at least one positive radial solution $(\bar{u}, \overline{w})$ such that $\bar{u}(y) \leqslant 2$ and $\overline{w}(y) \leqslant 2$ in $y \in \Omega$.

5.2. Continuous dependence of solutions on functional parameters in exterior domain

Let us consider the case when nonlinearities in system (26) depend on functional parameter $\mathbf{z} : [1, +\infty) \to R^s$, $\mathbf{z} \in \widetilde{\mathbf{Z}} \subset L^p((1, +\infty), R^s)$, $s \in N$, $p > 1$. Then (26) takes the form

$$-\operatorname{div}\big(l_i\big(\|y\|\big)v_i\big) = \widetilde{F}_i\big(\|y\|, \mathbf{v}, \mathbf{z}\big) \quad \text{in } \Omega, \text{ for all } i \in \{1, \dots, k\}, \tag{30}$$

with $\mathbf{v} := (v_1, \dots, v_k)$, $\widetilde{F} : (1, +\infty) \times \mathbf{J} \times R^s \to R$, $\widetilde{F}'_i = \frac{\partial \widetilde{F}}{\partial v_i}$, $i \in \{1, \dots, k\}$. Of course we are still studying solutions vanishing on the unit sphere and at infinity, namely satisfying (27) and (28). It is clear that we have to make conditions on the nonlinearities which guarantee that for each $\mathbf{z} \in \widetilde{\mathbf{Z}}$ function $(1, +\infty) \times \mathbf{J} \ni (t, \mathbf{u}) \mapsto \widetilde{F}(t, \mathbf{u}, \mathbf{z}(t))$ satisfies hypotheses (F1)–(F4). Thus, we assume that

(F1z) there exist k-dimensional interval $\mathbf{I} := [0, \tilde{d}_1] \times \cdots \times [0, \tilde{d}_k]$, $\tilde{d}_1, \dots, \tilde{d}_k \in R_+$, and a neighborhood $\mathbf{J}$ of $\mathbf{I}$ such that $\widetilde{F} \in C((1, +\infty) \times \mathbf{J} \times R^s, R)$ and function $\mathbf{J} \ni u \mapsto \widetilde{F}(t, \mathbf{u}, \mathbf{z}(t))$ is convex for each $\mathbf{z} \in \widetilde{\mathbf{Z}}$ and $t \in (1, +\infty)$ and $|\int_1^{+\infty} t^{n-1}\widetilde{F}(t, 0, \mathbf{z}(t))\,dt| < \infty$;

(F2z) for each $i \in \{1, \dots, k\}$, $(x, \mathbf{z}) \mapsto \widetilde{F}'_i(t, x, \mathbf{z})$ is continuous and nonnegative in $\mathbf{J} \times R^s$ for all $t \in (1, +\infty)$ and $\int_1^{+\infty} t^{n-1}\widetilde{F}'_i(t, 0, \mathbf{z}(t))\,dt \neq 0$;

(F4z) for all $i \in \{1, \dots, k\}$ there exists $\overline{\varphi}_i : (1, +\infty) \to R$ such that

$$\sup_{\mathbf{v} \in \mathbf{I}} F'_i\big(l, \mathbf{v}, \mathbf{z}(l)\big) \leqslant \overline{\varphi}_i(l) \quad \text{in } (1, +\infty)$$

for each $\mathbf{z} \in \widetilde{\mathbf{Z}}$ and $t \in (0, 1)$, functions $l \mapsto l^{2n-2+\frac{1-n}{q'}}\overline{\varphi}_i(l)$ belongs to $L^{q'_i}((1, +\infty), R)$ and $\int_1^{+\infty} l^{n-1}\overline{\varphi}_i(l)\,dl \leqslant 4(n-2)\,dc_i l_{\min}^2$ for a certain positive $d \leqslant \min_{i \in \{1, \dots, k\}} \tilde{d}_i$.

Remark 6. Let us note that complicated exponents in (F2) and (F4z) are associated with transformation of the problem with PDEs into the Dirichlet problem for ODEs.

Applying Theorem 9 we obtain the following results

Theorem 16. *Suppose that* (F1z)–(F2z), (F4z) *and* (F3) *hold and assume that the sequence of parameters* $\{\mathbf{z}_m\}_{m=1}^{\infty} \subset \widetilde{\mathbf{Z}}$ *is a.e. convergent to* $\mathbf{z}_0 \in \widetilde{\mathbf{Z}}$ *in* $[1, +\infty)$. *Let us denote by* $\bar{\mathbf{v}}_m \in C(\overline{\Omega}, R^k) \cap C^2(\Omega, R^k)$ *a radial positive solution of* (30), (27), (28) *corresponding to* $\mathbf{z} = \mathbf{z}_m$, $m \in \mathbf{N}$.

Then there exists a subsequence, still denoted by $\{\bar{\mathbf{v}}_m\}_{m=1}^{\infty}$, *uniformly convergent to a certain* $\bar{\mathbf{v}}_0$ *in* $\overline{\Omega}$ *which is the classical solution of* (30), (27), (28) *with* $\mathbf{z} = \mathbf{z}_0$. *Moreover* $\bar{\mathbf{v}}_0$ *is radial and* $\bar{\mathbf{v}}_0(y) \in \mathbf{I}$ *for all* $y \in \Omega$.

5.3. Continuous dependence of solutions on boundary conditions in exterior domain

Now we apply again Theorem 9 to investigate the continuous dependence of solutions for the system of PDEs on boundary conditions.

Theorem 17. *Suppose that* (F1z)–(F2z), (F3), (F4z) *with*

$$\mathbf{Z} := \{\tilde{\mathbf{c}}^m\}_{m=1}^{\infty} \cup \{\tilde{\mathbf{c}}^0\} \subset \underbrace{[0,+\infty) \times \cdots \times [0,+\infty)}_{k}$$

and suppose that the sequence of parameters $\{\tilde{\mathbf{c}}^m\}_{m=1}^{\infty}$ *tends to* $\tilde{\mathbf{c}}^0 := (\tilde{c}_1^0, \ldots, \tilde{c}_k^0)$. *Then for each* $m = 1, 2, \ldots,$ *problem*

$$\begin{cases} -\operatorname{div}\big(l_i(\|y\|)\nabla v_i\big) = F_i'(\|y\|, \mathbf{v}) & \text{for } y \in \Omega, \\ \lim\limits_{\|y\| \to \infty} v_i(y) = 0, \end{cases} \qquad \text{for all } i \in \{1, \ldots, k\} \tag{31}$$

($l_i \equiv 1$, $\mathbf{v} : \Omega \to R^k$, $\mathbf{v} = (v_1, \ldots, v_k)$) *with boundary condition*

$$v_i(y) = \tilde{c}_i^m \quad \text{for } \|y\| = 1, \text{ for all } i \in \{1, \ldots, k\} \tag{32}$$

possesses at least one radial positive solution $\bar{\mathbf{v}}^m \in C(\overline{\Omega}, R^k) \cap C^2(\Omega, R^k)$.

Moreover there exists a subsequence, still denoted by $\{\bar{\mathbf{v}}^m\}_{m=1}^{\infty}$, *uniformly convergent to a certain* $\bar{\mathbf{v}}^0$ *in* $\overline{\Omega}$ *which is the positive radial solution of* (31) *with boundary condition*

$$v_i(y) = \tilde{c}_i^0 \quad \text{for } \|y\| = 1. \tag{33}$$

Proof. It is clear that we may reduce problem (31)–(32), via suitable transformation (the same as in previous subsection) into (4) with nonlinearities $\widetilde{G}_i'$ given by

$$\widetilde{G}_i'(t, \mathbf{v}, \tilde{\mathbf{c}}) := \frac{1}{(n-2)^2}(1-t)^{\frac{2n-2}{2-n}} F_i'\big((1-t)^{\frac{1}{2-n}}, \mathbf{v} + (1-t)\tilde{\mathbf{c}}\big)$$

for all $i \in \{1, \ldots, k\}$ with $\tilde{\mathbf{c}} = (\tilde{c}_1, \ldots, \tilde{c}_k)$. By (F1z)–(F2z), (F3), (F4z) with $\mathbf{Z} := \{\tilde{\mathbf{c}}^m\}_{m=1}^{\infty} \cup \{\tilde{\mathbf{c}}^0\}$, we state that $\widetilde{G}_i'$ given above satisfies the assumptions of Remark 4 and Theorem 9. Thus we infer for each $m = 1, 2, \ldots$ the existence of at least one classical radial, positive solution $\bar{\mathbf{v}}^m$ for (31)–(32) and the uniform convergence of $\{\bar{\mathbf{v}}^m\}_{m=1}^{\infty}$ (up to a subsequence) to a certain $\bar{\mathbf{v}}^0$ in $\overline{\Omega}$ which is a classical solutions of (31)–(33). Moreover $\bar{\mathbf{v}}^0$ is radial and $\bar{\mathbf{v}}^0(y) \in I$. □

5.4. Continuous dependence of solutions on boundary conditions in annular domain

In this subsection we consider the following system of PDEs in annular domain

$$-\operatorname{div}\big(l_i(\|y\|)\nabla v_i\big) = F_i'(\|y\|, \mathbf{v}) \quad \text{for } y \in \Omega, \text{ for all } i \in \{1, \ldots, k\}, \tag{34}$$

$$v_i(y) = 0 \quad \text{for } \|y\| = r, \tag{35}$$

$$v_i(y) = \tilde{c}_i \quad \text{for } \|y\| = \mathbf{R}, \tag{36}$$

where $l_i \equiv 1$, $\Omega := \{y \in R^n, r < \|y\| < R\}$, $\tilde{\mathbf{c}} = (\tilde{c}_1, \ldots, \tilde{c}_k)$, $\tilde{c}_i \in [0, +\infty)$, $n \geqslant 3$. It is well-known fact that using the change of variable $t := h(l)$ given by

$$h(l) = -\frac{A}{l^{n-2}} + B, \quad \text{where } A := \frac{(r\mathbf{R})^{n-2}}{\mathbf{R}^{n-2} - r^{n-2}}, \; B := \frac{\mathbf{R}^{n-2}}{\mathbf{R}^{n-2} - r^{n-2}},$$

we transform the above system to (4) with nonlinearities $\widetilde{G}_i'$ given by

$$\widetilde{G}_i'(t,\mathbf{v},\tilde{\mathbf{c}}) := \frac{1}{(n-2)^2}\left(\frac{A}{(B-t)^{n-1}}\right)^{\frac{2}{n-2}} F_i'\left(\left(\frac{A}{B-t}\right)^{\frac{1}{n-2}}, \mathbf{v}+t\tilde{\mathbf{c}}\right) \quad \text{and} \quad l_i(t) = \dot{l}_i\left(\left(\frac{A}{B-t}\right)^{\frac{1}{n-2}}\right)$$

for all $i \in \{1,\dots,k\}$. As in the previous subsection we consider above problem under conditions which make nonlinearities $\widetilde{G}_i'$ satisfy assumptions of Theorem 9, namely

(F1′) there exist k-dimensional interval $\mathbf{I} := [0,\tilde{d}_1] \times \cdots \times [0,\tilde{d}_k]$, $\tilde{d}_1,\dots,\tilde{d}_k \in (\max_i \tilde{c}_i, +\infty)$, and a neighborhood $\mathbf{J}$ of $\mathbf{I}$ such that $\widetilde{F} \in C((r,R)\times \mathbf{J}, R)$ and $\mathbf{J} \ni \mathbf{u} \mapsto \widetilde{F}(t,\mathbf{u})$ is convex for all $t \in (r,\mathbf{R})$, $|\int_r^{\mathbf{R}} \widetilde{F}(l,(-\frac{A}{l^{n-2}}+B)\tilde{\mathbf{c}})\,dl| < \infty$;

(F2′) for each $i \in \{1,\dots,k\}$, $\mathbf{J} \ni x \mapsto \widetilde{F}_i'(t,x)$ is continuous and positive in $\mathbf{J}$ for all $t \in (r,\mathbf{R})$ and $\int_r^{\mathbf{R}} \widetilde{F}_i'(l,(-\frac{A}{l^{n-2}}+B)\tilde{\mathbf{c}})\,dl \neq 0$;

(F4′) for all $i \in \{1,\dots,k\}$ there exists $\overline{\varphi}_i \in L^{q'}((r,\mathbf{R}),\mathbf{R})$ such that

$$\sup_{\mathbf{v}\in\mathbf{I}} F_i'\left(l, \mathbf{v} + \left(-\frac{A}{l^{n-2}} + B\right)\tilde{\mathbf{c}}\right) \leqslant \overline{\varphi}_i(l) \quad \text{in } (1,+\infty)$$

and

$$\int_r^{\mathbf{R}} l^{n-1}\overline{\varphi}_i(l)\,dl \leqslant 4(n-2)Adc_i l_{\min}^2$$

for a certain positive $d \leqslant \min_{i\in\{1,\dots,k\}} \tilde{d}_i$.

As in the previous subsection Remark 4 and Theorem 9 lead to the existence of at least one positive radial classical solutions for parameters satisfying the above assumptions and the continuous dependence of solutions on boundary conditions.

Theorem 18. *Assume that* (F1′)–(F2′), (F3), (F4′) *with* $\mathbf{Z} := \{\tilde{\mathbf{c}}^m\}_{m=1}^{\infty} \cup \{\tilde{\mathbf{c}}^0\} \subset \underbrace{[0,+\infty)\times\cdots\times[0,+\infty)}_{k}$ *and suppose that the sequence of parameters* $\{\tilde{\mathbf{c}}^m\}_{m=1}^{\infty}$ *tends to* $\tilde{\mathbf{c}}^0 := (\tilde{c}_1^0,\dots,\tilde{c}_k^0)$. *Then for each* $m = 1,2,\dots,$ *problem* (34)–(36), *where the last boundary condition reads*

$$v_i(y) = \tilde{c}_i^m \quad \text{for } \|y\| = \mathbf{R}, \ \text{for all } i \in \{1,\dots,k\}$$

possesses at least one radial positive solution $\bar{\mathbf{v}}^m \in C(\overline{\Omega}, R^k) \cap C^2(\Omega, R^k)$.

Moreover there exists a subsequence, still denoted by $\{\bar{\mathbf{v}}^m\}_{m=1}^{\infty}$, *uniformly convergent to a certain* $\bar{\mathbf{v}}^0$ *in* $\overline{\Omega}$ *which is the positive radial classical solution of* (34)–(36) *where the last boundary condition reads*

$$v_i(y) = \tilde{c}_i^0 \quad \text{for } \|y\| = 1.$$

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